

FUNCTION THEORY OF ANTILINEAR OPERATORS

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Abstract. Unlike in complex linear operator theory, polynomials or, more generally, Laurent series in antilinear operators cannot be modelled with complex analysis. There exists a corresponding function space, though, surfacing in spectral mapping theorems. These spectral mapping theorems are inclusive in general. Equality can be established in the self-adjoint case. The arising functions are shown to possess a biradial character. It is shown that to any given set of Jacobi parameters corresponds a biradial measure yielding these parameters in an iterative orthogonalization process in this function space, once equipped with the corresponding L^2 structure.

Key words. antilinear operator, Laurent series, spectral mapping, biradial function, biradial measure, Jacobi operator, Hankel operator

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1. Introduction. Classical complex analysis is ubiquitous in complex linear Hilbert space operator theory; see, e.g., [15, 16, 1] and references therein. The interplay between these two fields has found many forms of which the spectral mapping theorem is the most well-known and undoubtedly most widely used. In real linear operator theory, complex linear operators constitute one extreme while antilinear operators constitute the other one. There is strong evidence to expect that these two extremes turn out to be mathematically equally rich [12, 13, 10, 11]. In this paper, a function space structure is identified which has an analogous connection with antilinear operators as complex analysis has with complex linear operators. This function space surfaces in spectral mapping theorems for antilinear operators. An L^2 theory for it arises in the iterative construction of invariant subspaces for self-adjoint antilinear operators once the appearing Jacobi operators are connected with biradial measure spaces.

For the spectral mapping theorems, suppose A is a bounded antilinear operator on a complex Hilbert space H . Antilinear means that $A\lambda = \bar{\lambda}A$ for any complex number λ . Then, for simplicity, take a polynomial $p(\lambda) = \sum_{k=0}^j \alpha_k \lambda^k$. If $x \in H$ is an eigenvector of A , there holds

$$p(A)x = \hat{p}(\lambda)x, \quad (1.1)$$

where p has transformed to $\hat{p}(\lambda) = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} (\alpha_{2k} + \alpha_{2k+1}\lambda) |\lambda|^{2k}$. More generally, by allowing more complicated analytic functions f , the transformed functions can be expressed as

$$\hat{f}(\lambda) = u(|\lambda|^2) + v(|\lambda|^2)\lambda.$$

with sufficiently regular complex valued functions u and v . These functions, denoted by $\mathcal{C}(r2)$, constitute a vector space over \mathbb{C} carrying a natural notion of product. The spectral mapping for the spectrum $\sigma(A)$ then takes the form

$$\hat{f}(\sigma(A)) \subset \sigma(f(A)). \quad (1.2)$$

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The equality cannot be established in general. However, if A is additionally self-adjoint, the equality is shown to hold in (1.2). In particular, when treated antilinearly, Hankel operators fit in this category in a natural way.

For the L^2 theory for these functions, even if x is not an eigenvector of A , collecting all the possible vectors on the left-hand side of (1.1) gives rise to an invariant subspace

$$K(A; x) = \overline{\{p(A)x \mid p \in \mathcal{P}\}}$$

of A , where \mathcal{P} denotes the set of polynomials. In the self-adjoint case, the function space $\mathcal{C}(r_2)$ arises again once A is represented on $K(A; x)$ with an antilinear Jacobi operator

$$J_{\#}\tau,$$

where $J_{\#}$ is a complex symmetric (typically infinite) matrix and τ denotes the conjugation operator. To characterize the Jacobi parameters on the diagonals of $J_{\#}$, a biradially supported L^2 theory for $\mathcal{C}(r_2)$ is devised. A curve in \mathbb{C} is said to be bi-radial if it intersects every origin centred circle at most at two points. Namely, it is shown that to any given set of bounded Jacobi parameters corresponds a compactly supported biradial measure yielding these parameters in an iterative orthogonalization process for polynomials \hat{p} in the respective L^2 space. The case of unbounded Jacobi parameters can be treated in terms of conditions on the moments recorded in a Hankel-like matrix. Regarding the lack of uniqueness of this correspondence, the finite dimensional case is completely solved.

The paper is organized as follows. In Section 2 spectral mapping theorems for antilinear operators are derived. The corresponding function space structure is identified. In Section 3 a theory for self-adjoint antilinear Jacobi operator is developed. To deal with the Jacobi parameter problem, biradial L^2 spaces are introduced. In Section 4 the unbounded case is considered. In Section 5 the non-injective determination of the Jacobi parameters is solved in finite dimensional cases.

2. Functions of antilinear operators and spectral theory. A continuous additive¹ operator on a complex Hilbert space H is real linear, i.e., it commutes with real scalars. This fact can be found already in Banach's classic book on linear operators [2].² Denote the family of such operators by $\mathcal{B}(H)$. The norm of $B \in \mathcal{B}(H)$ is defined as

$$\|B\| = \sup_{\|x\|=1} \|Bx\|.$$

The adjoint of B is the real linear operator B^* satisfying

$$(Bx, y)_{\mathbb{R}} = (x, B^*y)_{\mathbb{R}}$$

for every $x, y \in H$. Here $(\cdot, \cdot)_{\mathbb{R}} = \operatorname{Re}(\cdot, \cdot)$, where (\cdot, \cdot) denotes the inner product on H . If $B^* = B$, then B is said to be self-adjoint.

There exists a unique separation of B into its complex linear and antilinear parts as

$$B = C + A,$$

¹An operator B on H is additive if $B(x + y) = Bx + By$ for any $x, y \in H$.

²It is instructive to bear in mind that Banach's book started with additive operators and dealt only with real scalars, a fact which sometimes was regarded as curious [19, p.397].

where

$$C = \frac{1}{2}(B - iBi) \text{ and } A = \frac{1}{2}(B + iBi). \quad (2.1)$$

That is, then C is complex linear while A is antilinear, i.e., $A\lambda = \bar{\lambda}A$ holds for any $\lambda \in \mathbb{C}$.

Of course, the case of complex linear operators, i.e., when $A = 0$, has been extensively studied. In what follows, we are more interested in the antilinear case, i.e., when $C = 0$. To this end, the following space of polynomials was introduced in [12].

DEFINITION 2.1. *Polynomials of the form*

$$\hat{p}(\lambda) = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} (\alpha_{2k} + \alpha_{2k+1}\lambda) |\lambda|^{2k} \quad (2.2)$$

with $\alpha_k \in \mathbb{C}$ and, for j even $\alpha_{j+1} = 0$, are denoted by $\mathcal{P}_j(r2)$. Their union $\cup_{j=0}^{\infty} \mathcal{P}_j(r2)$ is denoted by $\mathcal{P}(r2)$.

For a nonzero polynomial, the greatest integer l such that $\alpha_l \neq 0$ in (2.2) is called the degree of the polynomial.

To see how these relate with real linear operators, take a standard analytic polynomial $p(\lambda) = \sum_{k=0}^j \alpha_k \lambda^k$. If $B \in \mathcal{B}(H)$, then $p(B)$ is defined as

$$p(B) = \sum_{k=0}^j \alpha_k B^k.$$

Of course, polynomials in complex linear operators appear regularly. Polynomials in antilinear operators have recently proved useful in numerical linear algebra and approximation theory [5, 11, 12]. A first notable difference between these two extremes is the fact that a polynomial in an antilinear operator typically becomes genuinely a real linear operator, i.e., it has nonzero complex and antilinear parts. Second, $\mathcal{P}_j(r2)$ is a natural function space to analyze polynomials in antilinear operators.

Namely, associate with p the polynomial \hat{p} defined in (2.2). At the most fundamental level, such a transformation occurs with a spectral mapping theorem for antilinear operators. This is readily seen in terms of eigenvalues. If A is antilinear and $Ax = \lambda x$ for a nonzero $x \in H$ and $\lambda \in \mathbb{C}$, then (1.1) holds. Consequently, the eigenvalues of A are mapped with \hat{p} to be among the eigenvalues of $p(A)$.

Definition 2.1 was not aimed at maximal generality, however. First, aside from eigenvalues, we have a more general notion of spectrum. Second, Laurent series in a complex linear operator can be defined as long as the spectrum is contained in the annulus of convergence. The spectrum of a real linear operator is defined in a natural way as follows.

DEFINITION 2.2. *The spectrum of $B \in \mathcal{B}(H)$ consists of those points $\lambda \in \mathbb{C}$ for which $\lambda I - B$ is not boundedly invertible. The set of these points is denoted by $\sigma(B)$.*

The spectrum of a real linear operator is always compact. For antilinear operators, it is circularly symmetric with respect to the origin, and it can be empty as well; see [13]. Hence, a lack of spectral radius means that spectrum alone is not sufficient to determine convergence of Laurent series in real linear operators. Still, if f is analytic in an annulus centred at the origin with the Laurent series

$$f(\lambda) = \sum_{k=-\infty}^{\infty} \alpha_k \lambda^k, \quad (2.3)$$

then, under appropriate assumptions on an antilinear A , the real linear operator $f(A)$ is well defined in terms of the series.³ (If $\alpha_k = 0$ for $k = -1, -2, \dots$, then we have a disc centred at the origin.) We assume that the spectrum $\sigma(A)$ is contained in the annulus of convergence. By inspecting eigenvalues, f gets transformed in the process completely analogously as

$$\hat{f}(\lambda) = \sum_{k=-\infty}^{\infty} (\alpha_{2k} + \alpha_{2k+1}\lambda) |\lambda|^{2k} = u(|\lambda|^2) + v(|\lambda|^2)\lambda. \quad (2.4)$$

Clearly, u and v are polynomials if and only if f is. Observe that \hat{f} maps origin centred circles to circles. In particular, we regard these functions as biradial for the following reason.

EXAMPLE 1. The functions u and v in (2.4) are “biradially” uniquely determined in the following sense. Suppose $\theta_1, \theta_2 \in [0, 2\pi)$ and $\theta_1 \neq \theta_2$. Then, for r fixed, the conditions

$$\begin{cases} u(r^2) + v(r^2)re^{\theta_1} &= a(r) \\ u(r^2) + v(r^2)re^{\theta_2} &= b(r) \end{cases} \quad (2.5)$$

with $a(r)$ and $b(r)$ given, determine the values of $u(r^2)$ and $v(r^2)$.⁴ Of course, if $v \equiv 0$, then we are dealing a standard radial function.

Denote by $\mathcal{C}(r2)$ functions of the form on right-hand side of (2.4). Assuming that the antilinear A is additionally self-adjoint, we shall allow continuous functions of this type by noting that A^2 is complex linear positive semidefinite. To ensure that this is consistent, let us invoke the following proposition.

PROPOSITION 2.3 ([13]). *Let $A \in \mathcal{B}(H)$ be antilinear. Then $\lambda \in \sigma(A)$ if and only if $|\lambda|^2 \in \sigma(A^2)$.*

DEFINITION 2.4. *Let $A \in \mathcal{B}(H)$ be antilinear and self-adjoint. Let u and v be complex valued continuous functions defined on the compact subset $\sigma(A^2) \subset [0, \infty)$. Then the function $\hat{f} : \sigma(A) \rightarrow \mathbb{C}$ defined by*

$$\hat{f}(\lambda) = u(|\lambda|^2) + v(|\lambda|^2)\lambda \quad (2.6)$$

is called a continuous biradial function. Define

$$f(A) = u(A^2) + v(A^2)A.$$

Clearly, $\mathcal{C}(r2)$ is a vector space over \mathbb{C} . There exists a natural notion of (non-commutative) product as well. To this end, consider again (1.1). For two elements $\hat{f}(\lambda) = u_1(|\lambda|^2) + v_1(|\lambda|^2)\lambda$ and $\hat{g}(\lambda) = u_2(|\lambda|^2) + v_2(|\lambda|^2)\lambda$ of $\mathcal{C}(r2)$, we have for the eigenvalues

$$f(A)g(A)x = \hat{h}(\lambda)x = (\hat{f} * \hat{g})(\lambda)x \quad (2.7)$$

once we define

$$\hat{h}(\lambda) = u_1(|\lambda|^2)u_2(|\lambda|^2) + |\lambda|^2 v_1(|\lambda|^2)\overline{v_2(|\lambda|^2)} + \left(u_1(|\lambda|^2)v_2(|\lambda|^2) + \overline{u_2(|\lambda|^2)}v_1(|\lambda|^2) \right) \lambda.$$

³Form $\limsup_{j \rightarrow \pm\infty} |\alpha_j|^{-1/j}$ and compare with $\limsup_{j \rightarrow \pm\infty} \|A^j\|^{1/j}$.

⁴This makes extending $\mathcal{C}(r2)$ apparent. To determine the values $2k$ -radially, we are lead to consider functions of the form $\sum_{j=0}^{2k-1} u_j(|\lambda|^{2k})\lambda^j$, where u_j are sufficiently smooth and $k \in \mathbb{N}$.

Or, in short

$$\hat{f} * \hat{g} = (u_1 + v_1 \lambda \tau)(u_2 + v_2 \lambda), \quad (2.8)$$

where τ denotes the conjugation operator.

After identifying an appropriate function space for antilinear operators, we are concerned with having a spectral mapping theorem. To this end we need the following lemma.

LEMMA 2.5. *Let $A \in \mathcal{B}(H)$ be antilinear and self-adjoint, and let $\hat{f} : \sigma(A) \rightarrow \mathbb{C}$ be a continuous biradial function. Then for any $\epsilon > 0$ there exists a polynomial p such that*

$$\|f(A) - p(A)\| < \epsilon \quad \text{and} \quad \max_{\lambda \in \sigma(A)} |\hat{f}(\lambda) - \hat{p}(\lambda)| < \epsilon.$$

Proof. The result follows by applying the Weierstrass approximation theorem to u and v in (2.6) on the compact set $\sigma(A^2)$ and using the continuous function theory for the complex linear A^2 . \square

THEOREM 2.6. *Let $A \in \mathcal{B}(H)$ be antilinear. Assuming $f(A)$ is defined for f in (2.3), there holds*

$$\hat{f}(\sigma(A)) \subset \sigma(f(A)).$$

Moreover, if A is self-adjoint, then this further holds for all continuous biradial functions $\hat{f} : \sigma(A) \rightarrow \mathbb{C}$ in (2.6).

Proof. For any $B \in \mathcal{B}(H)$ holds

$$\sigma_a(B) \cup \overline{\sigma_a(B^*)} = \sigma(B), \quad (2.9)$$

where $\sigma_a(B)$ denotes the approximate point spectrum of B and the bar denotes complex conjugation (not closure).

Take $\lambda \in \sigma_a(A)$. Then for any given $\epsilon > 0$ there exists a unit vector $x \in H$ such that $Ax = \lambda x + v$ with $\|v\| \leq \epsilon$. Let $\theta \in \mathbb{R}$. Since $Ae^{i\theta}x = e^{-i\theta}\lambda x + e^{-i\theta}v$, we may conclude that $\sigma_a(A)$ is circularly symmetric with respect to the origin. Therefore by (2.9), also $\sigma(A)$ is circularly symmetric with respect to the origin.

Consider a rational function $r(\lambda) = \sum_{k=-l}^j \alpha_k \lambda^k$ with $l, j \in \mathbb{N}$. (For this the identity (1.1) holds analogously.) By continuity, we have $r(A)x = \hat{r}(\lambda)x + w$, where $\|w\|$ can be made arbitrarily small by choosing ϵ small enough. Consequently, $\hat{r}(\lambda) \in \sigma_a(r(A))$ whenever $\lambda \in \sigma_a(A)$. Regarding f , we have $\|f(A) - r(A)\| \leq \hat{\epsilon}$ and $\max_{\mu \in \sigma(A)} |\hat{f}(\mu) - \hat{r}(\mu)| \leq \hat{\epsilon}$ for any $\hat{\epsilon} > 0$ by choosing l and j large enough. Thereby

$$\begin{aligned} \|f(A)x - \hat{f}(\lambda)x\| &\leq \|f(A)x - r(A)x\| + \|r(A)x - \hat{r}(\lambda)x\| + \|\hat{r}(\lambda)x - \hat{f}(\lambda)x\| \\ &\leq 2\hat{\epsilon} + \|w\|, \end{aligned}$$

and it follows that $\hat{f}(\lambda) \in \sigma_a(f(A))$. The case with self-adjoint A is similar, only then we have a polynomial in place of r obtained by applying Lemma 2.5.

Similarly can be dealt with the case $\lambda \in \overline{\sigma_a(A^*)}$. To see this, it suffices to consider the case of an eigenvalue of A^* and the corresponding circle of radius $|\lambda|$ centred at the origin. The claim follows from $r(A)^* = \sum_{k=-l}^j A^{*k} \overline{\alpha_k} = \sum_{k=-\lfloor \frac{j}{2} \rfloor}^{\lfloor \frac{j}{2} \rfloor} (\overline{\alpha_{2k}} + \alpha_{2k+1} A^*) A^{*2k}$ after multiplying by the corresponding eigenvector. \square

In general, the equality does not hold. For example, if the spectrum of A is empty, then with $\hat{f}(\lambda) = |\lambda|^2$ we have $\hat{f}(\sigma(A)) = \emptyset$, but $\sigma(A^2) \neq \emptyset$. The equality can be established in the following case, whose proof will be postponed until the end of Section 3.2.

THEOREM 2.7. *Assume $A \in \mathcal{B}(H)$ is antilinear and self-adjoint. Then, for all continuous biradial functions $\hat{f} : \sigma(A) \rightarrow \mathbb{C}$ in (2.6),*

$$\hat{f}(\sigma(A)) = \sigma(f(A)). \quad (2.10)$$

COROLLARY 2.8. *Assume $A \in \mathcal{B}(H)$ is antilinear and self-adjoint. Then*

$$\|f(A)\| = \max_{\lambda \in \sigma(A)} |\hat{f}(\lambda)| = \max_{\lambda \in \sigma(f(A))} |\lambda|.$$

For an illustration, Hankel operators yield an immediate nontrivial family of self-adjoint antilinear operators.

EXAMPLE 2. Hankel operators [17] constitute a natural family of self-adjoint antilinear operators once treated as follows. (See also [18, Sec. 7 and 8].) Denote by $H^2(\mathbf{D})$ the Hardy space, where \mathbf{D} is the unit disc. Let $a \in L^\infty(\mathbf{T})$, where \mathbf{T} denotes the unit circle and $P : L^2(\mathbf{T}) \rightarrow H^2(\mathbf{D})$ is the orthogonal projector onto $H^2(\mathbf{D})$.⁵ Then define

$$g \mapsto PM_a \bar{g}$$

on $H^2(\mathbf{D})$, where M_a is the multiplication operator $g \mapsto ag$. Represented on l^2 , we have $H\tau$, where H is an infinite Hankel matrix.⁶

Like Hankel operators, self-adjoint antilinear operators have also been studied in a complex linear setting more generally; see [7, 8] where also many examples are given.

We do not know how to completely characterize those antilinear operators for which the spectral mapping Theorem 2.6 holds with equality. However, we *conjecture* that these are precisely those antilinear A for which $\sigma(A^2) \subset [0, \infty)$. Necessity follows by taking $f(\lambda) = \lambda^2$ and using Proposition 2.3. In finite dimensional H , sufficiency is established by the notion of contriangularizability, which is equivalent to $\sigma(A^2) \subset [0, \infty)$ [9]. (For the probability of being contriangularizable, see [12].)

EXAMPLE 3. Suppose an antilinear $A \in \mathcal{B}(H)$ is such that $\sigma(A^2) \subset [0, \infty)$. Then also Gelfand's formula

$$\lim_{j \rightarrow \infty} \|A^j\|^{1/j} = \max_{\lambda \in \sigma(A)} |\lambda|$$

holds.

3. Antilinear Jacobi operators and L^2 theory for $\mathcal{C}(r2)$. Like in the complex linear case, the spectrum is intimately related with the notion of invariant subspace. That is, if an antilinear operator has an eigenvector, then its span yields an invariant subspace. By invariance is meant the following.

DEFINITION 3.1. *A subspace K of H is said to be invariant for an operator $B \in \mathcal{B}(H)$ if $BK \subset K$.*

⁵Recall that $H^2(\mathbf{D})$ can be identified with those elements of $L^2(\mathbf{T})$ which have vanishing Fourier coefficients for negative indices.

⁶This antilinear treatment leads to different problems. For instance, unitary similarity for $H\tau$ becomes unitary consimilarity for H . We are only aware of the unitary similarity problem for H [14].

For an antilinear $A \in \mathcal{B}(H)$ and $x \in H$, form the invariant subspace

$$K(A; x) = \overline{\{p(A)x \mid p \in \mathcal{P}\}}, \quad (3.1)$$

where \mathcal{P} denotes the set of polynomials. If x is an eigenvector, then $K(A; x)$ is one dimensional. For the other extreme, if $K(A; x) = H$, then x is said to be a cyclic vector for A , like in the complex linear case.

An orthonormal basis of $K(A; x)$ can be generated analogously to the finite dimensional case described in [5]. If the dimension of (3.1) is infinite, then A is represented as

$$H_{\#}\tau : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N}),$$

where the (infinite) matrix $H_{\#}$ is of Hessenberg type with real subdiagonal entries and τ denotes the standard conjugation operation on $l^2(\mathbb{N})$. If the dimension of (3.1) is finite, say n , replace $l^2(\mathbb{N})$ with \mathbb{C}^n . In case A is self-adjoint, $H_{\#}$ is a tridiagonal complex symmetric Jacobi matrix. In this way complex Jacobi operators arise naturally in antilinear operator theory.

PROPOSITION 3.2. *Let $A \in \mathcal{B}(H)$ be antilinear and self-adjoint, and let $x, y \in H$. If $y \perp K(A; x)$, then $K(A; y) \perp K(A; x)$.*

Proof. Let p and q be polynomials, and denote $q(\lambda) = u(\lambda^2) + v(\lambda^2)\lambda$, where u and v are polynomials. Then

$$\begin{aligned} (p(A)x, q(A)y) &= (p(A)x, u(A^2)y) + (p(A)x, v(A^2)Ay) \\ &= (\overline{u(A^2)}p(A)x, y) + \overline{(A\overline{v}(A^2)p(A)x, y)} \\ &= (\tilde{p}(A)x, y) + \overline{(\tilde{q}(A)x, y)} = 0, \end{aligned}$$

where we have used the fact that A^2 is \mathbb{C} -linear self-adjoint and \tilde{p}, \tilde{q} are some polynomials. \square

By Proposition 3.2, given a bounded self-adjoint antilinear operator A on H , we can express H as an orthogonal direct sum of invariant subspaces

$$H = \bigoplus_{\alpha} K(A; x_{\alpha}). \quad (3.2)$$

We denote the restriction of A to these subspaces by

$$A_{\alpha} = A|_{K(A; x_{\alpha})}. \quad (3.3)$$

We have

$$\sigma(A) = \overline{\bigcup_{\alpha} \sigma(A_{\alpha})}, \quad (3.4)$$

since $\sigma(A^2) = \overline{\bigcup_{\alpha} \sigma(A_{\alpha}^2)}$, and we apply Proposition 2.3 noting that $\sigma(A)$ and $\sigma(A_{\alpha})$ are circularly symmetric with respect to the origin.

With these preliminaries, next we show that there exists a correspondence between self-adjoint antilinear Jacobi operators and positive measures on the plane, much as in complex linear operator theory. The construction relies on L^2 theory for $\mathcal{C}(r2)$ by orthogonalizing polynomials (2.2) supported on these measure spaces. Since the invariant subspaces $K(A; x)$ may be finite or infinite dimensional, we shall deal with them separately, starting with the finite dimensional case. Then, once we have handled the infinite dimensional case, Theorem 2.7 will be proved.

It is instructive to bear in mind that, as described in Example 2, everything applies to Hankel operators.

3.1. Biradial measures with finite support and antilinear Jacobi operators on finite dimensional spaces. In what follows it is shown, by partly following [12], that there exists a correspondence between (discrete) biradial measures and antilinear Jacobi operators. For convenience, the following notation for the monomials appearing in Definition 2.1 is employed.

DEFINITION 3.3. Let $k \in \mathbb{N}$. Denote by $\langle \lambda \rangle^k$ the monomials defined by

$$\langle \lambda \rangle^k = \begin{cases} \lambda |\lambda|^{2j}, & \text{if } k \text{ is odd and } k = 2j + 1, \\ |\lambda|^k, & \text{if } k \text{ is even.} \end{cases}$$

To put it short, biradiality means the second item in the following definition.

DEFINITION 3.4. Assume $n \in \mathbb{N}$. Let $\rho_k > 0$, $\lambda_k \in \mathbb{C}$ ($k = 1, \dots, n$) be such that

- (i) $\sum_{k=1}^n \rho_k = 1$,
- (ii) $\lambda_1, \dots, \lambda_n$ are distinct and any origin centred circle intersects at most two of them.

Furthermore, the complex numbers $\lambda_1, \dots, \lambda_n$ are assumed to be ordered such that

$$\begin{aligned} |\lambda_{2k-1}| &= |\lambda_{2k}| & \text{for } k = 1, \dots, m, \\ |\lambda_{2k-1}| &< |\lambda_{2k+1}| & \text{for } k = 1, \dots, m-1, \\ |\lambda_{2m+1}| &< |\lambda_{2m+2}| < \dots < |\lambda_n| \end{aligned}$$

for $m \in \mathbb{N}$. Then a positive measure on the plane defined by

$$\rho = \sum_{k=1}^n \rho_k \delta_{\lambda_k} \quad (3.5)$$

is called a biradial measure with finite support.

Assume ρ is a biradial measure with finite support. On $\mathcal{P}(r2)$ let us use the L^2 inner product

$$\langle p, q \rangle = \int_{\mathbb{C}} p \bar{q} d\rho = \sum_{k=1}^n p(\lambda_k) \overline{q(\lambda_k)} \rho_k. \quad (3.6)$$

Consider the monomials $1, \langle \lambda \rangle^1, \langle \lambda \rangle^2, \dots, \langle \lambda \rangle^{n-1}$. Executing the Gram-Schmidt orthogonalization process with respect to this the inner product yields an orthonormal sequence of polynomials p_0, p_1, \dots, p_{n-1} such that each p_j has degree j . This is a consequence of the following proposition guaranteeing that $\|p\|^2 = \langle p, p \rangle > 0$ for all $p \in \mathcal{P}(r2)$ with degree less than n .

PROPOSITION 3.5 ([12]). Let $p \in \mathcal{P}_d(r2)$ be nonzero. The following claims hold:

1. If p has two distinct zeroes of the same modulus, then all numbers of that modulus are zeroes.
2. Let m be the number of nonzero moduli for which all numbers of that modulus are zeroes and let s be the number of moduli for which exactly one number is a zero. Then $2m + s \leq d$.

Consider the product (2.8). Expressing the polynomial $\lambda \tau p_j(\lambda) = \overline{\lambda p_j(\lambda)}$ as a linear combination of p_0, \dots, p_{j+1} by imposing orthogonality gives rise to the three term recurrence

$$\beta_{j+1} p_{j+1}(\lambda) = \overline{\lambda p_j(\lambda)} - \alpha_{j+1} p_j(\lambda) - \beta_j p_{j-1}(\lambda), \quad (j = 0, \dots, n-2), \quad (3.7)$$

where $p_{-1}(\lambda) \equiv 0$, $\alpha_{j+1} = \langle \lambda \overline{p_j}, p_j \rangle$ and $\beta_{j+1} = \langle \lambda \overline{p_j}, p_{j+1} \rangle$. Observe that $\beta_{j+1} > 0$ since the leading term of p_{j+1} has positive coefficient, a consequence of executing the Gram-Schmidt process. Proposition 3.5, the L^2 space $(\mathcal{P}_n(r^2), \langle \cdot, \cdot \rangle)$ is n dimensional. Therefore $\lambda \overline{p_{n-1}(\lambda)}$ is a linear combination of p_0, \dots, p_{n-1} and hence $\beta_n = 0$. In this way we have associated these so-called called Jacobi parameters $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $\beta_1, \dots, \beta_{n-1} > 0$ with the given biradial measure ρ .

To express this linear algebraically, let $Q = (q_{kj}) \in \mathbb{C}^{n \times n}$ be the unitary matrix with columns j defined by

$$q_{kj} = \sqrt{\rho_k} p_{j-1}(\lambda_k), \quad (k = 1, \dots, n). \quad (3.8)$$

Then, by using the recursion (3.7), we get

$$D_{\#} \overline{Q} = Q J_{\#}, \quad (3.9)$$

where

$$J_{\#} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & \alpha_{n-1} & \beta_{n-1} \\ 0 & \cdots & 0 & \beta_{n-1} & \alpha_n \end{bmatrix}, \quad (3.10)$$

$$D_{\#} = \text{diag}(\lambda_1, \dots, \lambda_n). \quad (3.11)$$

In this way, given a biradial measure with finite support, we have the corresponding complex Jacobi matrix (3.10).

Suppose, conversely, that we are given a complex Jacobi matrix (3.10) and we want to find a corresponding biradial measure. We start with the following proposition.

PROPOSITION 3.6. *Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and let $\beta_1, \dots, \beta_{n-1} > 0$. Then there exist numbers $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and a unitary matrix $Q \in \mathbb{C}^{n \times n}$ with positive first column such that the equation (3.9) is satisfied, when $J_{\#}$ and $D_{\#}$ are defined by formulae (3.10) and (3.11). Furthermore, the numbers $\lambda_1, \dots, \lambda_n$ satisfy the property (ii) in Definition 3.4.*

Proof. By [9, Corollary 4.4.4], there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ and a nonnegative diagonal matrix $\Sigma_{\#} = \text{diag}(\sigma_1, \dots, \sigma_n)$ such that

$$\Sigma_{\#} \overline{U} = U J_{\#}. \quad (3.12)$$

We show that no three of $\sigma_1, \dots, \sigma_n$ can be equal. To reach a contradiction, we assume, without loss of generality, that $\sigma_1 = \sigma_2 = \sigma_3$. Let $V \in \mathbb{C}^{3 \times n}$ be the first three rows of U and let v_1, \dots, v_n be the columns of V . By (3.12), we have the recursion

$$\begin{aligned} \sigma_1 \overline{v_1} &= \beta_1 v_2 + \alpha_1 v_1, \\ \sigma_1 \overline{v_{j+1}} &= \beta_{j+1} v_{j+2} + \alpha_{j+1} v_{j+1} + \beta_j v_j, \quad j = 1, \dots, n-2. \end{aligned}$$

It follows that each v_j is a linear combination of v_1 and $\overline{v_1}$. This implies that the rank of V is at most two, contradicting the fact that the rank is three.

Let $\Theta \in \mathbb{R}^{n \times n}$ be a diagonal matrix such that $Q = e^{i\Theta} U$ has a nonnegative first column. Define $D_{\#} = e^{2i\Theta} \Sigma_{\#}$. Then the identity (3.9) holds. That all $\lambda_1, \dots, \lambda_n$

are distinct in (3.11) can be shown by an argument similar to the one given above. Furthermore, if q_j is a column of Q^* , then $J_{\#}\overline{q_j} = \lambda_j q_j$ and it is easy to see that the first entry of q_j must be nonzero. \square

COROLLARY 3.7. *Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and let $\beta_1, \dots, \beta_{n-1} > 0$. Then there exists a biradial measure μ with support of cardinality n such that the complex Jacobi matrix corresponding to μ is given by (3.10).*

Proof. Let $Q = (q_{kj}) \in \mathbb{C}^{n \times n}$ and $D_{\#} \in \mathbb{C}^{n \times n}$ be according to the previous proposition and let $\rho_k = q_{k1}^2$. Define the measure $\rho = \sum_{k=1}^n \rho_k \delta_{\lambda_k}$. For each $j = 1, \dots, n$ we then regard (3.8) as a system of equations with p_{j-1} as the unknown. These systems have invertible Vandermonde-type matrices, by Proposition 3.5, and are therefore uniquely solvable. A glance at the equation (3.9) then shows that the polynomials p_{j-1} satisfy the recursion (3.7) with the same coefficients as the given complex Jacobi matrix $J_{\#}$. \square

To sum up, based on constructing an L^2 theory for $\mathcal{C}(r2)$ and then orthogonalizing polynomials (2.2) according to (3.7), we have shown that the mapping from biradial measures with finite support to complex Jacobi matrices of type given in Corollary 3.7 is surjective. It is not injective, however. This lack of injectivity will be completely described in Section 5, after establishing surjectivity in the infinite dimensional case.

3.2. Biradial measures with infinite support and antilinear Jacobi operators on infinite dimensional spaces. We now turn to biradial measures with infinite supports in preparation to the study of antilinear Jacobi operators on infinite dimensional Hilbert spaces. We denote $\mathbb{R}^+ = [0, \infty)$.

DEFINITION 3.8. *A Borel probability measure ρ on \mathbb{C} is said to have finite moments, if it satisfies*

$$\int_{\mathbb{C}} |\lambda|^n d\rho(\lambda) < \infty \quad \text{for all } n = 0, 1, 2, \dots \quad (3.13)$$

DEFINITION 3.9. *Let ρ_1^+, ρ_2^+ be positive Borel measures on \mathbb{R}^+ such that*

$$\rho_1^+(\mathbb{R}^+) + \rho_2^+(\mathbb{R}^+) = 1.$$

Let ϕ_1, ϕ_2 be real-valued Borel-measurable functions on \mathbb{R}^+ and define the measurable transformation $R_{\phi_j} : \mathbb{C} \rightarrow \mathbb{C}$ by

$$R_{\phi_j}(z) = e^{-i\phi_j(|z|)} z.$$

Define the positive Borel measures ρ_j on \mathbb{C} by

$$\rho_j(E) = \rho_j^+(R_{\phi_j}(E) \cap (\mathbb{R}^+ \times \{0\})), \quad (j = 1, 2)$$

where $E \subset \mathbb{C}$ is Borel-measurable. Then $\rho = \rho_1 + \rho_2$ is called a biradial measure. Moreover, it is called symmetric if $\phi_1(r) = \phi_2(r) + \pi$ (modulo 2π).

We can represent a biradial measure ρ in an equivalent form as follows. Let $\rho^+ = \rho_1^+ + \rho_2^+$ and let a_1, a_2 be the Radon-Nikodym derivatives $a_j = d\rho_j^+/d\rho^+$ ($j = 1, 2$). Note that $0 \leq a_1(r), a_2(r) \leq 1$ and $a_1(r) + a_2(r) = 1$ for almost every r . Suppose $E \subset \mathbb{C}$ is measurable and write as follows (where we use $R_{\phi_j}(E)$ as short-hand for $R_{\phi_j}(E) \cap (\mathbb{R}^+ \times \{0\})$ with the vacuous second coordinate removed)

$$\begin{aligned} \rho_j(E) &= \int_0^\infty \chi_{R_{\phi_j}(E)} d\rho_j^+ = \int_0^\infty a_j(r) \delta_r(R_{\phi_j}(E)) d\rho^+(r) \\ &= \int_0^\infty a_j(r) \delta_{re^{i\phi_j(r)}}(E) d\rho^+(r). \quad (j = 1, 2) \end{aligned}$$

Hence we can write

$$\rho(E) = \int_0^\infty \mu_r(E) d\rho^+(r), \quad \text{where} \quad \mu_r(E) = \sum_{j=1}^2 a_j(r) \delta_{re^{i\phi_j(r)}}(E). \quad (3.14)$$

The formula (3.14) is the disintegration of the measure ρ with respect to the measurable map $T: \mathbb{C} \rightarrow \mathbb{R}^+$, $T(z) = |z|$, and ρ^+ [4]. Conversely, we could define a measure by the formula (3.14) and rewrite it in the form of Definition 3.9.

EXAMPLE 4. Let μ be a positive Borel measure on \mathbb{R} . Define the measures ρ_1^+, ρ_2^+ on \mathbb{R}^+ by $\rho_1^+(E) = \mu(E)$ and $\rho_2^+(E) = \mu(-E \cap (-\infty, 0))$, where $E \subset \mathbb{R}^+$. Let $\phi_1 \equiv 0$ and $\phi_2 \equiv \pi$. Then the measure ρ on \mathbb{C} in Definition 3.9 is supported inside $\mathbb{R} \times \{0\}$, and $\rho(F \times \{0\}) = \mu(F)$ for all Borel sets $F \subset \mathbb{R}$.

EXAMPLE 5. Let ρ_1^+ be a Borel probability measure with support \mathbb{R}^+ and absolutely continuous with respect to the Lebesgue measure. Let $\{A_k\}_{k=1}^\infty$ be a partition of \mathbb{R}^+ into metrically dense subsets, i.e., for all k we have A_k Borel measurable and $\rho_1^+(A_k \cap (x - \epsilon, x + \epsilon)) > 0$ for all $x \in \mathbb{R}^+, \epsilon > 0$ (e.g. [6]). Let $\{q_k\}_{k=1}^\infty$ be an enumeration of the rationals and define $\phi_1 = \sum_{k=1}^\infty q_k \chi_{A_k}$. Let $\rho_2^+ = 0$. Then the measure ρ in Definition 3.9 satisfies $\rho(B(z, r)) > 0$ for every open disc $B(z, r) \subset \mathbb{C}$ and therefore the support of ρ is \mathbb{C} .

With respect to the monomials $\langle \lambda \rangle^k$, measures on \mathbb{C} can be reduced in dimension as follows.

LEMMA 3.10. *Let μ be a Borel (probability) measure on \mathbb{C} with finite moments. Then there exists a symmetric biradial measure ρ such that*

$$\int_{\mathbb{C}} p d\mu = \int_{\mathbb{C}} p d\rho \quad \text{for all } p \in \mathcal{P}(r^2). \quad (3.15)$$

Moreover, if μ is compactly supported, then also ρ is.

Proof. Define the measurable map $T: \mathbb{C} \rightarrow \mathbb{R}^+$, $T(z) = |z|$ and let $\{\mu_r\}_{r \in \mathbb{R}^+}$ be the disintegration of μ with respect to T and the image measure $\rho^+ = \mu T^{-1}$ [4]. For nonnegative integer k , we then have

$$\int_{\mathbb{C}} \langle \lambda \rangle^k d\mu(\lambda) = \int_0^\infty \int_{|\lambda|=r} \langle \lambda \rangle^k d\mu_r(\lambda) d\rho^+(r),$$

where, for almost all r , μ_r is a Borel probability measure supported in $\{|\lambda| = r\}$. For odd k , write $k = 2l + 1$, and then we get

$$\int_{\mathbb{C}} \langle \lambda \rangle^k d\mu(\lambda) = \int_0^\infty \int_{|\lambda|=r} \lambda d\mu_r(\lambda) r^{2l} d\rho^+(r),$$

and we denote the inner integral (center of mass) on the right-hand side by $C(r)$. Then we can choose $a_1(r), a_2(r)$ such that $0 \leq a_1(r), a_2(r) \leq 1$ and $a_1(r) + a_2(r) = 1$, together with $\phi_1(r), \phi_2(r)$, such that

$$C(r) = \sum_{j=1}^2 a_j(r) r e^{i\phi_j(r)}.$$

If $C(r) \neq 0$ is on the circle $\{|\lambda| = r\}$, we can set $a_2(r) = 0$ and then $\phi_1(r)$ is uniquely determined (modulo 2π). If $C(r) \neq 0$ is inside the circle, these choices are non-unique

as illustrated in Figure 5.1, and we let $\phi_2(r) = \phi_1(r) + \pi$ (modulo 2π) to get symmetry. If $C(r) = 0$, we let $\phi_1(r) = 0$, $\phi_2(r) = \pi$. We now define the measures

$$\nu_r = \sum_{j=1}^2 a_j(r) \delta_{re^{i\phi_j(r)}}.$$

(Note that if $C(r) \neq 0$, our choices for $a_1(r)$, $a_2(r)$, $\phi_1(r)$ and $\phi_2(r)$ make ν_r the unique measure of this form.) We observe that

$$\int_{|\lambda|=r} \lambda d\mu_r(\lambda) = C(r) = \int_{|\lambda|=r} \lambda d\nu_r(\lambda).$$

The measure defined by $\rho(E) = \int_0^\infty \nu_r(E) d\rho^+(r)$ is symmetric biradial, see formula (3.14), and we now have

$$\int_{\mathbb{C}} \langle \lambda \rangle^k d\mu(\lambda) = \int_{\mathbb{C}} \langle \lambda \rangle^k d\rho(\lambda) \quad (3.16)$$

for all odd k . It is easy to see that (3.16) is true for even k as well, and then (3.15) follows.

The final claim follows from the fact that ρ^+ is compactly supported if μ is. \square

PROPOSITION 3.11. *Let μ be a compactly supported biradial measure. Then the set of polynomials $\mathcal{P}(r2)$ is dense in $L^2(\mu)$.*

Proof. Let $R > 0$ be such that the support of μ is contained in an origin-centred closed disc of radius R . We denote the disintegration of μ as in formula (3.14), and denote $\lambda_1(r) = re^{i\phi_1(r)}$, $\lambda_2(r) = re^{i\phi_2(r)}$.

Take $f \in L^2(\mu)$ and $\epsilon > 0$. Let g be a compactly supported smooth function on \mathbb{C} such that $\|f - g\|_{L^2(\mu)}^2 < \epsilon/2$. For r such that $\lambda_1(r) = \lambda_2(r)$, let $u(r^2) = g(\lambda_1(r))$ and $v(r^2) = 0$, and otherwise let $u(r^2)$ and $v(r^2)$ be the unique numbers such that

$$g(\lambda_j(r)) = u(r^2) + v(r^2)\lambda_j(r), \quad (j = 1, 2).$$

For r such that $\lambda_1(r) \neq \lambda_2(r)$, we now have

$$\frac{g(\lambda_1(r)) - g(\lambda_2(r))}{\lambda_1(r) - \lambda_2(r)} = v(r^2),$$

where the left hand side is bounded. Hence v is a bounded function and it follows that u is bounded as well. Let p and q be ordinary polynomials such that

$$\int_0^R |u(r^2) - p(r^2)|^2 d\rho^+(r) < \frac{\epsilon}{8} \quad \text{and} \quad \int_0^R |v(r^2)r - q(r^2)r|^2 d\rho^+(r) < \frac{\epsilon}{8}.$$

Then we have

$$\int_{\mathbb{C}} |f(\lambda) - p(|\lambda|^2) - q(|\lambda|^2)\lambda|^2 d\mu(\lambda) < \epsilon.$$

\square

With these measure theoretic preparations, let ρ be a Borel (probability) measure on \mathbb{C} with finite moments. On $\mathcal{P}(r2)$, we define the inner product

$$\langle p, q \rangle = \int_{\mathbb{C}} p\bar{q} d\rho. \quad (3.17)$$

We then apply the Gram-Schmidt process to the monomials $1, \langle \lambda \rangle^1, \langle \lambda \rangle^2, \dots$ with respect to the inner product (3.6) and obtain an orthonormal sequence of polynomials p_0, p_1, p_2, \dots , where p_j has degree j . The process breaks down if and only if $\dim \mathcal{P}_n(r2) \leq n$ for some n . In the breakdown case, for the least such n , we have the orthonormal polynomials p_0, \dots, p_{n-1} and the corresponding Jacobi parameters $\{\alpha_j\}_{j=1}^n, \{\beta_j\}_{j=1}^{n-1}$ similar to the case of biradial measures with finite support. If the process does not break down, we get infinitely many Jacobi parameters $\{\alpha_j\}_{j=1}^\infty$ with positive $\{\beta_j\}_{j=1}^\infty$ and the recursion (3.7) holds for all $j = 0, 1, 2, \dots$. These Jacobi parameters are recorded in the infinite matrix

$$J_\# = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots \\ \beta_1 & \alpha_2 & \beta_2 & \ddots \\ 0 & \beta_2 & \alpha_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \quad (3.18)$$

PROPOSITION 3.12. *Let ρ, ρ' be Borel (probability) measures with finite moments and at least n Jacobi parameters, $\{\alpha_j\}_{j=1}^n, \{\beta_j\}_{j=1}^{n-1}$ and $\{\alpha'_j\}_{j=1}^n, \{\beta'_j\}_{j=1}^{n-1}$, respectively. Then $\alpha_j = \alpha'_j$ ($j = 1, \dots, n$) and $\beta_j = \beta'_j$ ($j = 1, \dots, n-1$) if and only if*

$$\int_{\mathbb{C}} \langle \lambda \rangle^k d\rho = \int_{\mathbb{C}} \langle \lambda \rangle^k d\rho' \quad \text{for all } k = 0, 1, \dots, 2n-1. \quad (3.19)$$

Proof. The proof for the real Jacobi parameter case carries over easily [21, Proposition 1.3.4]. \square

The following is a version of Favard's theorem for bounded antilinear Jacobi operators. We prove it similarly as in [21].

THEOREM 3.13. *Let $\{\alpha_j\}_{j=1}^\infty, \{\beta_j\}_{j=1}^\infty$ be bounded Jacobi parameters. Then there exists a compactly supported symmetric biradial measure μ on \mathbb{C} such that the Jacobi parameters corresponding to μ are $\{\alpha_j\}_{j=1}^\infty, \{\beta_j\}_{j=1}^\infty$.*

Proof. By Corollary 3.7, for each n there exists a biradial measure μ_n such that the corresponding Jacobi parameters are $\{\alpha_j\}_{j=1}^n, \{\beta_j\}_{j=1}^{n-1}$, and we denote the corresponding complex Jacobi matrix by $J_\#^{[n]}$. Since the n points supporting μ_n are coneigenvalues of $J_\#^{[n]}$, the support of μ_n is contained in the closed disc $\overline{B}(0, \|J_\#^{[n]}\|) \subset \overline{B}(0, \|J_\# \tau\|) =: B$. Moreover, $\mu_n(\mathbb{C}) = 1$ is bounded for all n and therefore there exists a weakly converging subsequence $\{\mu_{n_j}\}_{j=1}^\infty$ such that

$$\lim_{j \rightarrow \infty} \int_B f d\mu_{n_j} = \int_B f d\mu$$

for all continuous functions f on B , where μ is a Borel probability measure supported in B . For each nonnegative integer k , we choose $f(\lambda) = \langle \lambda \rangle^k$ and use Proposition 3.12 to conclude that the Jacobi parameters of μ are $\{\alpha_j\}_{j=1}^\infty, \{\beta_j\}_{j=1}^\infty$. Moreover, an application of Lemma 3.10 replaces μ with a symmetric biradial measure. \square

We next obtain a spectral theorem for bounded antilinear self-adjoint operators. For an approach using spectral integrals, see [20].

THEOREM 3.14. *Let A be a bounded antilinear self-adjoint operator on a Hilbert space H . Suppose there exists a cyclic vector of A . Then there exists a compactly*

supported symmetric biradial measure μ on \mathbb{C} , and a \mathbb{C} -linear isometric isomorphism $U : H \rightarrow L^2(\mu)$, such that

$$UAU^{-1} = \lambda\tau, \quad (3.20)$$

where $\lambda\tau$ denotes the multiplication operator $f(\lambda) \mapsto \lambda\overline{f(\lambda)}$ on $L^2(\mu)$.

Proof. Let $x \in H$ be the cyclic vector and let $\{q_j\}_{j=1}^\infty$ be the orthonormal basis of H generated by the Arnoldi process starting from the vector x . Moreover, let $J_\#$ be the generated complex (infinite) Jacobi matrix with parameters $\{\alpha_j\}_{j=1}^\infty, \{\beta_j\}_{j=1}^\infty$. By Theorem 3.13 the measure μ exists, with orthonormal polynomials $\{p_j\}_{j=0}^\infty$, such that

$$\beta_j p_{j-1}(\lambda) + \alpha_{j+1} p_j(\lambda) + \beta_{j+1} p_{j+1}(\lambda) = \lambda \overline{p_j(\lambda)} \quad (j = 0, 1, 2, \dots),$$

where $p_{-1} \equiv 0$. The set of polynomials $\{p_j\}_{j=0}^\infty$ is an orthonormal basis of $L^2(\mu)$ by Proposition 3.11. We define the isometric isomorphism U by $U(q_j) = p_{j-1}$ ($j = 1, 2, \dots$) and the claim follows. \square

COROLLARY 3.15. *Under the assumptions of Theorem 3.14, for any continuous biradial function \hat{f} defined on $\sigma(A)$,*

$$Uf(A)U^{-1} = u(|\lambda|^2) + v(|\lambda|^2)\lambda\tau,$$

where u and v are as in (2.6).

Proof. Apply Lemma 2.5. \square

PROPOSITION 3.16. *Biradial measures μ given by Theorem 3.14 are supported in the spectrum $\sigma(A)$.*

Proof. By Theorem 3.14, we have $UA^2U^{-1} = |\lambda|^2$ and therefore

$$Up(A^2)U^{-1} = p(|\lambda|^2) \quad (3.21)$$

for all ordinary complex analytic polynomials p . The operator A^2 is \mathbb{C} -linear self-adjoint and positive semidefinite.

Let $\lambda_0 \notin \sigma(A)$. Let $g(\lambda) = f(|\lambda|^2)$, where f is a nonnegative compactly supported function on \mathbb{R}^+ such that f vanishes on $\sigma(A^2)$ and $g(\lambda_0) = 1$, which is possible due to Proposition 2.3. Let $\{p_j\}$ be a sequence of ordinary polynomials converging uniformly to f on the support of f . Then $\|f(A^2) - p_j(A^2)\| \rightarrow 0$ as $j \rightarrow \infty$, and, since $f(A^2) = 0$, from (3.21) we have

$$\int_{\mathbb{C}} g(\lambda) d\mu(\lambda) = 0.$$

Hence there exists an open set $V \subset \mathbb{C}$ containing λ_0 such that $\mu(V) = 0$ and therefore λ_0 is not in the support of μ . \square

We are now ready to prove Theorem 2.7 and Corollary 2.8.

Proof. of Theorem 2.7. The inclusion \subset for (2.10) is Theorem 2.6. For the inclusion \supset , take $\lambda_0 \notin \hat{f}(\sigma(A))$, and let H and A be decomposed as in (3.2) and (3.3). By Corollary 3.15, we have

$$U(\lambda_0 I_\alpha - f(A_\alpha))U^{-1} = \lambda_0 - u(|\lambda|^2) - v(|\lambda|^2)\lambda\tau. \quad (3.22)$$

Since $\sigma(A)$ is a compact set,

$$c := \min_{\lambda \in \sigma(A)} \left| |\lambda_0 - u(|\lambda|^2)| - |v(|\lambda|^2)\lambda| \right| = \min_{\lambda \in \sigma(A)} |\lambda_0 - \hat{f}(\lambda)| > 0,$$

and, since $\hat{f}(\sigma(A)) \supset \hat{f}(\sigma(A_\alpha))$ by (3.4), we have

$$0 < c \leq \min_{\lambda \in \sigma(A_\alpha)} \left| |\lambda_0 - u(|\lambda|^2)| - |v(|\lambda|^2)\lambda| \right| \quad \text{for all } \alpha.$$

Hence the multiplication operator on the right-hand side of equation (3.22) is boundedly invertible, with the inverse having the norm estimate

$$\max_{\lambda \in \sigma(A_\alpha)} \frac{1}{\left| |\lambda_0 - u(|\lambda|^2)| - |v(|\lambda|^2)\lambda| \right|} \leq \frac{1}{c} < \infty \quad \text{for all } \alpha.$$

We have shown that the operators $\lambda_0 I_\alpha - f(A_\alpha)$ are boundedly invertible for all α , with the norms of their inverses uniformly bounded. Hence $\lambda_0 I - f(A)$ is boundedly invertible, so $\lambda_0 \notin \sigma(f(A))$. \square

Proof. of Corollary 2.8. The second equality follows from Theorem 2.7. For the first, let H and A be decomposed as in (3.2) and (3.3), and note that $\|f(A)\| = \sup_\alpha \|f(A_\alpha)\|$. Then, by Corollary 3.15,

$$\begin{aligned} \|f(A_\alpha)g\|^2 &\leq \int_{\sigma(A_\alpha)} (|u(|\lambda|^2)| + |v(|\lambda|^2)\lambda|)^2 |h(\lambda)|^2 d\mu(\lambda) \\ &\leq \sup_{\lambda \in \sigma(A_\alpha)} |\hat{f}(\lambda)|^2 \|h\|_{L^2(\mu)}^2, \end{aligned}$$

where we denoted $h = Ug$. Hence $\|f(A_\alpha)\| \leq \sup_{\lambda \in \sigma(A_\alpha)} |\hat{f}(\lambda)|$. The inequality in the other direction is established by considering invertibility of $\lambda I_\alpha - f(A_\alpha) = \lambda(I_\alpha - \lambda^{-1}f(A_\alpha))$ via the Neumann series expansion in the usual way. \square

Starting from [3], multiplication operators have played a major role in understanding the \mathbb{C} -linear invariant subspace problem. For the corresponding problems in the antilinear case, the role of multiplication is taken by the operator (3.20).

4. Unbounded antilinear Jacobi operators and the moment problem.

Let ρ be a biradial measure with finite moments and define

$$m_k = \int_{\mathbb{C}} \langle \lambda \rangle^k d\rho(\lambda), \quad k = 0, 1, 2, \dots \quad (4.1)$$

In the corresponding moment problem we are given an arbitrary sequence of complex numbers $\{m_k\}_{k=1}^\infty$ such that, after a possible scaling, $m_0 = 1$. Of course, then necessarily $m_{2k} \geq 0$ for $k \in \mathbb{N}$. The problem consists of finding a biradial measure ρ such that the equation (4.1) is satisfied. The answer can be given in terms of the matrix

$$M = \begin{bmatrix} m_0 & \overline{m_1} & m_2 & \overline{m_3} & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \\ m_2 & \overline{m_3} & m_4 & \overline{m_5} & \cdots \\ m_3 & m_4 & m_5 & m_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (4.2)$$

where, denoting again the conjugation operator by τ , $M_{ij} = \tau^{j-1} m_{i+j-2}$. Observe that M is Hermitian, as well as, in a certain sense, a Hankel-like matrix. Thereby it is natural to contrast this with the classical Hamburger moment problem.

THEOREM 4.1. *Let $\{m_k\}_{k=1}^\infty \subset \mathbb{C}$ and $m_0 = 1$. Then there exists a biradial measure ρ with finite moments such that (4.1) holds if and only if the submatrix $M_{1:k,1:k}$ of (4.2) is positive semi-definite for all k .*

Proof. That the claim holds with a Borel probability measure μ , in lieu of a biradial measure ρ , follows from [22, Theorem 1]. This measure μ can then be replaced by a biradial measure ρ by Lemma 3.10. \square

COROLLARY 4.2. *The mapping from biradial measures to complex Jacobi parameters is surjective.*

Proof. Let $\{\alpha_j, \beta_j\}_{j=1}^\infty \in (\mathbb{C} \times (0, \infty))^\infty$ be complex Jacobi parameters. Define a sequence of polynomials $\{p_j\}_{j=0}^\infty$ by the recursion (3.7) starting with $p_0 \equiv 1$. Define an inner product in $\mathcal{P}(r2)$ by

$$\langle p, q \rangle = \sum_{j \geq 0} a_j \overline{b_j}, \quad (p, q \in \mathcal{P}(r2)),$$

where $p = \sum_j a_j p_j$ and $q = \sum_j b_j p_j$, and let

$$m_k = \langle \langle \lambda \rangle^k, 1 \rangle \quad \text{for } k = 0, 1, 2, \dots$$

From the recursion (3.7) we see that multiplication by $\lambda\tau$ in $\mathcal{P}(r2)$ is represented by $J_\# \tau$ in the basis $\{p_j\}_{j=0}^\infty$, where $J_\#$ is the infinite complex Jacobi matrix given by formula (3.18). From this we find

$$m_{i+j} = e_1^T (J_\# \tau)^{i+j} e_1 = \tau^j ((J_\# \tau)^j e_1)^* ((J_\# \tau)^i e_1) = \tau^j \langle \langle \lambda \rangle^i, \langle \lambda \rangle^j \rangle,$$

where $e_1 = [1 \ 0 \ 0 \ \dots]^T$. Since the inner product is positive-definite, the condition of Theorem 4.1 is satisfied and therefore there exists a biradial measure ρ such that

$$\langle \langle \lambda \rangle^k, 1 \rangle = m_k = \int_{\mathbb{C}} \langle \lambda \rangle^k d\rho \quad \text{for all } k.$$

Since $\tau^j \langle \lambda \rangle^{i+j} = \langle \lambda \rangle^i \overline{\langle \lambda \rangle^j}$, we get $\langle p, q \rangle = \int_{\mathbb{C}} p \overline{q} d\rho$ for any polynomials p and q . This completes the proof. \square

5. Noninjectivity of the mapping of measures to Jacobi operators. The mapping of biradial measures to antilinear Jacobi operators is surjective, but non-injective (even for measures with bounded support). We do not discuss this in full generality, but content ourselves with a precise characterization in the finite dimensional case covered in Section 3.1.

THEOREM 5.1. *Let ρ, ρ' be biradial measures with supports of cardinality n . Then ρ and ρ' have the same Jacobi parameters if and only if (using the notation of Definition 3.4) we have*

$$\begin{aligned} m &= m', \\ \rho_k &= \rho'_k, & \text{when } 2m+1 \leq k \leq n, \\ \rho_{2k-1} + \rho_{2k} &= \rho'_{2k-1} + \rho'_{2k}, & \text{when } 1 \leq k \leq m, \\ \lambda_k &= \lambda'_k, & \text{when } 2m+1 \leq k \leq n, \\ |\lambda_{2k}| &= |\lambda'_{2k}|, & \text{when } 1 \leq k \leq m, \\ \rho_{2k-1} \lambda_{2k-1} + \rho_{2k} \lambda_{2k} &= \rho'_{2k-1} \lambda'_{2k-1} + \rho'_{2k} \lambda'_{2k}, & \text{when } 1 \leq k \leq m. \end{aligned} \quad (5.1)$$

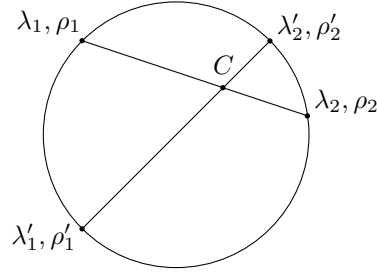


FIG. 5.1. An illustration of the case where the centre of mass C resides inside the circle. Here ρ_i, ρ'_i are the masses of the points and λ_i, λ'_i are their positions. The total mass and the centre of mass are unique.

Proof. It is easy to verify that the conditions (5.1) imply (3.19). Hence sufficiency follows from Proposition 3.12.

To prove necessity, assume the condition (3.19) holds. First, the equation (3.9) implies that $J_{\#} \overline{J_{\#}}$ and $D_{\#} \overline{D_{\#}}$ have the same eigenvalues. Hence the sets $\{|\lambda_i|\}_{i=1}^n$ and $\{|\lambda'_i|\}_{i=1}^n$ are the same, including multiplicities, which implies the first and the fifth condition in (5.1). Moreover, the conditions given by (3.19) for even k then imply the second and the third condition in (5.1). Finally, for odd k , (3.19) imply the fourth and the sixth condition in (5.1). \square

The preceding theorem says that the total mass on each origin centred circle is unique and the centre of the masses on each such circle is unique as well. If the centre of mass is on the circle, then all the mass is concentrated on a single unique point. If the centre of mass is inside the circle, then the mass is distributed among two non-unique points as illustrated in Figure 5.1.

There is also an immediate application of this result in numerical analysis.

EXAMPLE 6. Assume that $M_{\#} \in \mathbb{C}^{n \times n}$ is a very large matrix and $b \in \mathbb{C}^n$. The \mathbb{R} -linear GMRES (Generalized Minimal Residual) method is an iterative method for solving the linear system

$$M_{\#} \overline{x} = b.$$

In [12] its convergence behaviour was analyzed. If $M_{\#}$ complex symmetric, then it can be assumed that $M_{\#}$ is actually a diagonal matrix. Suppose, moreover, that $M_{\#}$ has distinct diagonal entries such that each origin centred circle intersected either two of them or none, and that $b \in \mathbb{R}^n$ is a vector with all its entries ones. Then in [12, Section 5] it was observed that the numerical convergence behaviour was such that the residual dropped only at every other iteration step.

We can now explain this observation as follows. The norm of the residual vector $r_k = b - M_{\#} \overline{x_k}$, where x_k is the approximation at the k th step, satisfies

$$\|r_k\| = \min_{\substack{p \in \mathcal{P}_k(r_2) \\ p(0)=1}} \|p(M_{\#})b\| \leq \min_{\substack{p \in \mathcal{P}_k(r_2) \\ p(0)=1}} \max_{\lambda \in \sigma(M_{\#})} |p(\lambda)| \|b\|.$$

By Theorem 5.1, there exists a unitary matrix $Q \in \mathbb{C}^{n \times n}$ such that $\rho = Q^* b \in \mathbb{R}^n$ and that the matrix $D_{\#} = Q^* M_{\#} \overline{Q}$ is a diagonal matrix with the property that if $\lambda \in \mathbb{C}$ appears as a diagonal entry then also $-\lambda$ does. Now we have

$$\|r_k\| = \min_{\substack{p \in \mathcal{P}_k(r_2) \\ p(0)=1}} \|p(D_{\#})\rho\| \leq \min_{\substack{p \in \mathcal{P}_k(r_2) \\ p(0)=1}} \max_{\lambda \in \sigma(D_{\#})} |p(\lambda)| \|\rho\|.$$

If $p(\lambda) = u(|\lambda|^2) + v(|\lambda|^2)\lambda$ is the minimizing polynomial to the problem on the right-most side, then due to symmetry, also $u(|\lambda|^2) - v(|\lambda|^2)\lambda$ is. Hence, by uniqueness, $v = 0$ and $p(\lambda) = u(|\lambda|^2)$. Thereby we may expect that the residual drops only at every other iteration step.

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